Journal of Applied Mathematics and Mechanics 72 (2008) 660–668

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Controlled rolling of a disc along a plane curve☆

L.D. Akulenko

Moscow, Russia

ARTICLE INFO

Article history:
Received 12 April 2007

ABSTRACT

The plane-parallel rolling of a disc along a fairly smooth curve under the action of perturbing and controlling forces and moments of forces is investigated. Models of the controlled motion corresponding to an explicit or implicit coordinate specification of the curve are constructed. The requirements imposed on the values of the sliding friction force and the normal pressure force, which ensure that the disc rolls without slipping and without loss of contact are determined. The problem of bringing the disc into the required state of motion in a time-optimal way by the action of the force of gravity and a limited axial moment of forces are considered. The situation when the reference curve contains a linear component (inclined upwards or downwards) and a periodic (wavy) component is investigated. Problems of optimal synthesis for a relatively large and relatively small controlling moment are considered.Investigations of the rolling of bodies of different geometrical shape on surfaces, both flat and curved (see Refs 1-4, etc.), are important for applications in modern machine construction, transport, mobile robot systems,[5–7] etc. However, there are no results of systematic investigations of the problems of the dynamics and control of rolling of bodies on curved surfaces, including comparatively simple bodies (cylinders, discs and spheres). From the applied point of view, the problem of controlled rolling of a disc under gravitational forces on a curve containing a linear incline and periodic smooth changes in the slope is of interest as a model of the motion of a wheel along a path with a curvilinear profile. The solution of problems of the control of the motion of a disc along a closed curve, for example, inside or outside a circle, surmounting obstacles or "rolling out" from hollows of different shapes etc. is of fundamental importance.

1. Formulation of the problem

A circular uniform disc of radius a is in contact with a plane smooth curve at a certain point of the curve (see Fig. 1). Regular rolling of the disc along the curve (without losing contact and without slipping) is assumed to be possible under the action of external and internal forces and moments of forces. The disc and the curve are assumed to be absolutely rigid and rough, and rolling friction is ignored.

To describe the motion of the disc it is necessary to introduce a sufficiently suitable system of coordinates, including a Cartesian system XY. We will assume as given the equation of the curve and expressions for the main geometrical characteristics

$$K(x, y) = 0, \quad K'^2_x + K'^2_y \neq 0, \quad x_1 \le x \le x_2, \quad y_1 \le y \le y_2; \quad \operatorname{tg}\beta = y' = -K'_x/K'_y$$

$$K(x, y) = 0, \quad K'^2_x + K'^2_y \neq 0, \quad x_1 \le x \le x_2, \quad y_1 \le y \le y_2; \quad \operatorname{tg}\beta = y' = -K'_x/K'_y$$

$$\kappa = \rho^{-1} = (K'^2_x + K'^2_y)^{-3/2} \begin{Vmatrix} K''_{xx} & K''_{xy} & K'_x \\ K''_{yx} & K''_{yy} & K'_y \\ K'_x & K'_y & 0 \end{Vmatrix} \tag{1.1}$$

Here K is a fairly smooth function in the region considered, the quantities x_i and y_i (i = 1, 2) can be bounded or infinite, β is the angle of inclination of the tangent at the point r = (x,y) of the curve while the function κ is the curvature of the curve and ρ is the radius of curvature. The disc can be in contact with the curve on one side or the other (conventionally under or above the curve).

☆ *Prikl. Mat. Mekh*. Vol. 72, No. 6, pp. 912–924, 2008.

E-mail address: kumak@ipmnet.ru.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2009.01.011

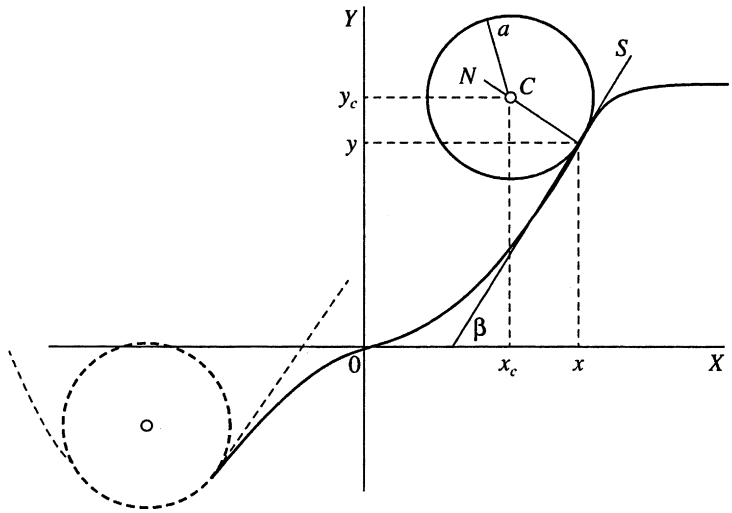

Fig. 1.

To simplify the description of the motion we will assume that, in the region considered, the curve can be represented uniquely in the form

$$y = y(x), \quad -\infty < x_1 \le x \le x_2 < \infty$$
$$\operatorname{tg}\beta = y'(x), \quad |y'| \le k^* < \infty, \quad |\beta| \le \operatorname{arctg} k^* < \frac{\pi}{2}$$
$$\kappa = y''(x)(1+y'^2(x))^{-3/2}, \quad |\kappa| \le \kappa^*, \quad \rho = \kappa^{-1} \tag{1.2}$$

The prime denotes a derivative with respect to the argument x.

Suppose, to be specific, the disc is situated above the curve (see Fig. 1). When it is rolling, as can easily be established, it is situated on one side of the curve since the slope of the curve is founded, according to (1.2). To avoid several (two or more) contact points and possible "impacts", we must impose certain requirements on the local curvature $\kappa(x)$, which will be described below.

In the case of regular rolling, the contact point moves along the curve (1.2) and travels a path (taking the sign into account), described by the quadrature

$$s(x) = s^0 + \int_{x^0}^{x} (1+y'^2(\zeta))^{1/2} d\zeta, \quad |s - s^0| \ge |x - x^0|$$
$$s'(x) = (1+y'^2(x))^{1/2} \ge 1, \quad \cos\beta = \frac{1}{s'}, \quad \sin\beta = y'/s' \tag{1.3}$$

Without loss of generality we can put $s^0 = x^0 = 0$. The function $s(x)$ is necessary in order to describe the angular motion of the disc, due to rolling. We can conveniently describe the angle of inclination of the tangent to the curve at the point $r = (x, y(x))$ by means of the expression $s'(x)$. Note that the geometrical centre C of the disc is on a straight line passing through the contact point orthogonal to the tangent, i.e., the following equalities hold

$$y'(Y-y) = X - x; \quad X = x_c, \quad Y = y_c, \quad (x_c - x)^2 + (y_c - y)^2 = a^2; \quad y = y(x)$$
$$x_c = x - a\sin\beta = x - a\frac{y'}{s'}; \quad y_c = y + a\cos\beta = y + \frac{a}{s'}, \quad s' = (1+y'^2)^{1/2} \tag{1.4}$$

To construct the equations of motion it is necessary to determine the relative and absolute rotation of the disc about the z axis or the axis passing through its centre. The angle of complete rotation (x) can be represented in the form

$$\psi(x) = \varphi(x) + \Delta\beta(x), \quad \varphi(x) = -s(x)/a, \quad \Delta\beta = \beta(x) - \beta(0), \quad \beta = \operatorname{arctg} y'(x) \tag{1.5}$$

The value of φ is determined by the length of the path $s(x)$ (1.3), and the addition $\Delta\beta$ is due to the change in the slope of the curve $y(x)$ (1.2).

Below, the variable x is used as the required generalized coordinate $x = x(t)$. To derive the equations of motion and the conditions for regular rolling due to the action of potential (gravitational) and other forces and moments of forces of various kinds (controlling, dissipative, etc.), we will calculate the vector of the velocity of motion of the centre of mass and absolute angular velocity of motion about the centre of mass. We will use relations (1.2)–(1.5) between the variables x and x_c, y_c and β.

2. Kinematic characteristics of the rolling of the disc

We will determine the components of the absolute velocity of the geometrical centre of the disc $\dot{r}_c = (\dot{x}_c, \dot{y}_c)$. Differentiating the expressions for x_c and y_c given by (1.4), we have

$$\dot{x}_c = \dot{x} - a\dot{\beta}\cos\beta = (1 - a\kappa)\dot{x}, \quad x = x(t)$$
$$\dot{y}_c = \dot{y} - a\dot{\beta}\sin\beta = y'(1 - a\kappa)\dot{x} = y'\dot{x}_c \tag{2.1}$$

It can be seen that the functions $\dot{x}_c, \dot{y}_c$ will be continuous in t if the function $\kappa(x)$ is continuous, i.e., the function $y(x)$ is twice continuously differentiable. We will assume the function $x(t)$ (2.1) to be fairly smooth.

We will calculate the absolute angular velocity of rotation of the disc about the axis passing through its geometrical centre. Differentiating $\psi(x)$ (1.5), we have

$$\dot{\psi} = \dot{\varphi} + \dot{\beta} = -a^{-1}(1 - a\kappa)s'\dot{x}, \quad \dot{\varphi} = -s'a^{-1}\dot{x} = -(1 + y'^2)^{1/2}a^{-1}\dot{x}$$
$$\dot{\beta} = y''(1 - y'^2)^{-1}\dot{x} = \kappa s'\dot{x}; \quad x = x(t) \tag{2.2}$$

For the function $\dot{\psi}$ to be continuous with respect to t when $\dot{x}(t)$ is continuous it is necessary for $y''(x)$ to be continuous with respect to x. This property is similar to that described above for $\dot{x}_c, \dot{y}_c$ (2.1). Hence, if the values of $y(x)$ are smoothly matched at certain points x_i, i.e., the derivatives $y'(x)$ are identical (for example, $y \sim |x - x_i|(x - x_i)$), but may have a discontinuity, expressions (2.1) and (2.2) may contain singularities – finite discontinuities (jumps) of the functions y'' and κ. When deriving the equation of motion and the conditions of regularity, expressions for $\ddot{x}_c, \ddot{y}_c$ and $\ddot{\psi}$ are required (see below), which leads to the functions $y'''(x)$, which will be singular of the Dirac δ-function type. If the piecewise-smooth profile $y(x)$ can have discontinuities, the degree of non-smoothness of the expressions for the derivatives is increased: the functions y'' and κ will be singular, which leads to δ-functions in the expressions for $\dot{x}_c, \dot{y}_c$ (2.1) and $\dot{\psi}$, $\dot{\beta}$ (2.2); the function $\dot{\varphi}$ remains piecewise-continuous, i.e., it has finite discontinuities. The calculation of the second derivatives with respect to t requires calculations of the function $y'''(x)$, the order of singularity of which in the case under consideration will be higher (the derivative of the δ-function). It is extremely difficult to investigate essentially non-linear differential equations containing singular dependences on the unknown function. The established properties of motion are unexpected and have not been mentioned in the scientific literature.

Using expressions (2.1), we calculate the required second derivatives $\ddot{x}_c, \ddot{y}_c$; we obtain

$$\ddot{x}_c = (1 - a\kappa)\ddot{x} - a\kappa'\dot{x}^2, \quad \kappa' = y'''/s'^{3/2} - 3y''^2y'/s'^{5/2}$$
$$\ddot{y}_c = y'(1 - a\kappa)\ddot{x} + [y''(1 - a\kappa) - ay'\kappa']\dot{x}^2, \quad y'' = \kappa s'^{3/2} \tag{2.3}$$

The presence of a derivative of the function of the curvature κ' requires the existence of a third derivative $y'''(x)$, which must be sufficiently smooth. Similar effects occur after calculating the angular acceleration of the disc

$$\ddot{\psi} = -(1 - a\kappa)s'a^{-1}\ddot{x} - (1 - a\kappa)s''a^{-1}\dot{x}^2 + \kappa s'\dot{x}^2$$
$$s'' = \frac{y''y'}{s'} = \kappa y's'^2 \tag{2.4}$$

Note that expressions $\dot{x}_c, \dot{y}_c$ (2.1) and $\dot{\psi}$ (2.2) contain the factor $(1 - a\kappa)$ in front of the derivative $\dot{x}$. This indicates a singularity in the motion of such a system when $\kappa = a^{-1}$. A similar coefficient is present in the expressions for $\ddot{x}_c, \ddot{y}_c$ (2.3) and $\ddot{\psi}$ (2.4). It is natural to assume that the curvature $\kappa(x)$ is small at all points of the curve $y(x)$, i.e., $y''(x)$ is small, so that the inequality $\kappa < a^{-1}$ holds. If $\kappa \le 0$ (convexity upwards), the established difficulty does not arise. Hence, the properties of the mechanical system considered (the smoothness of the profile and boundedness of the curvature) indicate its interesting specific features, not previously considered.

3. The equations of controlled rolling of a disc

When deriving the equation of motion in the Lagrange form[8] we will use expressions (2.1) for the components $\dot{x}_c, \dot{y}_c$ of the vector of the velocity $\dot{r}_c$ of the centre of mass and (2.2) for the angular velocity $\dot{\psi}$ of rotation of the disc. In order to simplify the expressions we will assume that the centre of mass of the disc and its geometrical centre C coincide. We will denote by m the mass of the disc and by I the moment of inertia about the axis passing through the centre of mass. The total kinetic energy of motion of the disc is due to the absolute velocity $\dot{r}_c$ of motion of the centre of mass and the total angular velocity $\dot{\psi}$ about the centre of mass; we have

$$T = T_m + T_I, \quad T_m = m(\dot{x}_c^2 + \dot{y}_c^2)/2, \quad T_I = I\dot{\psi}^2/2 \tag{3.1}$$

As follows from expressions (2.1) and (2.2), the terms and sum of the quantities (3.1) are homogeneous quadratic functions of the generalized velocity $\dot{x}$. The generalized inertial characteristic μ of the disc, taking relation (1.4) and expression (1.5) into account, depends on the x coordinate and the reduced mass m^*. Substituting expressions (2.1) and (2.2) into relations (3.1) we obtain the following representation of the kinetic energy $T(x, \dot{x})$, which is suitable for the use of the method of constructing Lagrange's equation,

$$T = \mu(x)\dot{x}^2/2, \quad \mu = m^*(1 - a\kappa)^2s'^2, \quad m^* = m + Ia^{-2} \tag{3.2}$$

Here κ and s' are functions of the variable x, according to relations (1.2) and (1.3).

We will write the potential energy V in general form, using the standard procedure,[8] and, in particular, for a uniform mass-force field (force of gravity)

$$V(x) = W(\mathbf{r}_c, \psi), \quad (W = W(x_c, y_c, \varphi, \beta))$$
$$V(x) = W_g(y_c) = mgy_c(x) + \text{const}, \quad y_c = y + a(1 + y'^2)^{-1/2} \tag{3.3}$$

Using expressions (3.2) and (3.3) we calculate the Lagrange function $T-V$ in terms of the Cartesian coordinate x and the derivative $\dot{x}$. Using standard methods we find the required equation of motion. The other generalized forces of an arbitrary physical nature can be taken into account in addition to the potential forces and the moments of the forces. Expressions for the elementary work δA of these forces on virtual displacements δr_c, $\delta\psi$ or δs, $\delta\psi$ are represented in the form[8]

$$\delta A = p_x\delta x + p_y\delta y + q\delta\psi = (p_x + y'p_y)\delta x + qb\delta x \equiv f\delta x$$
$$\delta A = p\delta s + q\delta\psi = ps'\delta x + qb\delta x \equiv b\delta x, \quad b(x) = (\kappa - a^{-1})s' \tag{3.4}$$

In addition to the expressions for the elementary work of the moments of the external forces on virtual rotations $q\delta\psi$ one can take into account the quantities $q_\varphi\delta\varphi$ and $q_\beta\delta\beta$ separately. The functions $p_{x,y}, q, p, f$ introduced in (3.4) may depend on the variables t, x, $\dot{x}$, and also on the control action u, depending on the purposes of the motion. As in Ref. 7 we obtain

$$\mu(x)\ddot{x} + \mu'(x)\frac{\dot{x}^2}{2} + V'(x) = f(t, x, \dot{x}, u) \tag{3.5}$$

Equation (3.5) describes the rolling of the disc in a regular mode without loss of contact and without slipping. We will determine the corresponding tangential force S and normal force N acting on the disc at the contact point.[7] Using equations of motion of the centre of mass similar to (2.3), we obtain, by differentiating the velocities $\dot{x}_c$, $\dot{y}_c$,

$$m\ddot{x}_c \equiv m(1 - a\kappa)\ddot{x} - ma\kappa'\dot{x}^2 = \frac{S}{s'} - \frac{Ny'}{s'} + G_x + P_x$$
$$m\ddot{y}_c \equiv my'(1 - a\kappa)\ddot{x} + m[y''(1 - a\kappa) - ay'\kappa']\dot{x}^2 = \frac{Sy'}{s'} + \frac{N}{s'} + G_y + P_y$$
$$\left(\mathbf{G} = -\frac{\delta W}{\delta \mathbf{r}_c};\ \mathbf{G} = (0, -mg), \quad V = W_g\right) \tag{3.6}$$

The system of linear algebraic equations (3.6) enables the unknowns S and N to be determined uniquely for the specified remaining quantities. This leads to the expressions investigated earlier for the cases of a horizontal straight line and an inclined straight line[2,5–7]

$$S = ms'(1 - a\kappa)\ddot{x} + m[y's'^2\kappa(1 - a\kappa) - a\kappa's']\dot{x}^2 - G_s - P_s$$
$$N = ms'^2\kappa(1 - a\kappa)\dot{x}^2 - G_n - P_n \tag{3.7}$$

The tangential components G_s and P_s and the normal components G_n and P_n of the forces are obtained from $G_{x,y}$ and $P_{x,y}$ by a similar transformation (by rotating the vectors $\mathbf{G}$ and $\mathbf{P}$ by an angle $-\beta$). The moments of the forces have an indirect effect via the variables x, $\dot{x}$, $\ddot{x}$. By (3.7) the quantity N is independent of the acceleration $\ddot{x}$. Moreover, it can be eliminated from the expression for S by means of Eq. (3.5). The requirement of regular rolling under natural conditions reduces to inequalities, which can be established a priori

$$N > 0, \quad |S| \le k_f N, \quad k_f \sim 1 \quad (k_f > 0) \tag{3.8}$$

Here k_f is the coefficient of dry friction according to the Amonton-Coulomb law.[2]

Equation of motion (3.5) allows of the formulation of standard control and optimization problems.[9] To be specific, suppose it is required to translate the phase point x, $\dot{x}$ into the state x^f, $\dot{x}^f$ by the instant of time $t_f < \infty$ and minimize the performance index J

$$x(t_0) = x^0, \quad \dot{x}(t_0) = \dot{x}^0, \quad x(t_f) = x^f, \quad \dot{x}(t_f) = \dot{x}^f, \quad J[u] \to \min_u, \quad u \in U \tag{3.9}$$

The time-optimal problem[7,9] with $J = t_f$, when the supporting curve is described by a smooth periodic function and also has a slope, contains rises and/or depressions, etc., is of interest for applications. Some examples are given below.

4. Examples of controlled rolling of a disc

We will take as examples, shapes of the supporting curve corresponding to actual route profiles. Periodic profiles with a slope (upwards or downwards) or without a slope are of interest for applications. We will consider the rolling of a disc in a uniform gravitational field under the action of a control moment of the force applied to its axis.

1°. We will first consider an inclined wavy profile of the form

$$y = y(x) = kx + h(1-\cos\lambda x), \quad \mathrm{tg}\beta = y' = k + \xi\sin\lambda x$$

$$s = s(x) = \int_0^x [1+y'^2(l)]^{1/2}dl, \quad \xi = k\lambda, \quad k, h, \lambda = \text{const}$$

$$s' = s'(x) = [1+y'^2(x)]^{1/2}, \quad \kappa = \kappa(x) = \xi\lambda\cos\lambda x/s'^3 \tag{4.1}$$

The parameters k, h and δ have a clear geometrical meaning: k is the coefficient of the average slope, h is the amplitude of the periodic component, λ is the "wave number" and $2\pi/\lambda$ is the wavelength. By relations (4.1) the value of $y'(x)$ is bounded for all $|x| < \infty$, i.e. $|\beta| < \pi/2$. The values of the function $|\beta| < \frac{\pi}{2}$ and have an upper bound. The curvature $\kappa(x)$ is a periodic sign-varying function with extrema at the points $x_n = \nu_n/\lambda$, where ν_n are the roots of the trigonometric equation

$$\sin\nu[1+(k+\xi\sin\nu)^2] + 3\xi\cos^2\nu(k+\xi\sin\nu) = 0$$

$$\nu_n = \pi n - 3\cos\pi n\xi k(1+k^2)^{-1/2} + O(\xi^2), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\kappa_n^{\pm} = \lambda\xi\cos\pi n(1+k^2)^{-3/2} + O(\xi^2); \quad |\lambda\xi|(1+k^2)^{-3/2} < a^{-1} \tag{4.2}$$

Note that when $k = 0$ (there is no average slope) the points of the extrema $\nu_n = \pi n$, and the quantities $\kappa_n^{\pm} = \pm\lambda\xi$; they must satisfy the condition $|\lambda\xi| < a^{-1}$. Otherwise, when rolling occurs under natural conditions, impact interactions of the edge of the disc and the profile of the route occur (see the dashed lines in Fig. 1).

We will investigate the equation of the controlled motion of the disc (3.5), corresponding to curve (4.1), in a potential field of mass forces with potential energy W_g (3.3), for example, a heavy disc when there is only a constrained control moment of the forces q in expressions (3.4). This equation reduces to the form

$$\mu(x)\ddot{x} + \mu'(x)\frac{\dot{x}^2}{2} + mg(1-a\kappa)y' = -q(1-a\kappa)a^{-1}s'$$

$$\mu = m^*s'^2(1-a\kappa)^2, \quad |q| \le q_0, \quad |a\kappa| < 1 \tag{4.3}$$

The functions $y'(x)$, $s'(x)$ and $\kappa(x)$ are defined by relations (4.1), and the parameter q_0 defines the constraint on the control moment q (3.4). Note that expression (4.1) for $s(x)$ has not so far been used, which considerably simplifies the description of the system, since all the functions are represented in an elementary analytical form. The function $s(x)$ is expressed in terms of elliptic integrals. In particular, when $k = 0$ or $|k| = |\xi|$ the expression of the function for $s(x)$ (4.1) reduces to complete and incomplete elliptic integrals of the second kind (see formula 4.12 below)

When $\xi = 0$ the supporting curve has the form of an inclined straight line $y = kx$, where $k > 0$ or $k < 0$; $\kappa(x) \equiv 0$. Then

$$s' = (1+k^2)^{1/2} = \text{const}, \quad \mu = m^*s'^2, \quad \mu' \equiv 0$$

The controlled system (4.3) reduces to the form

$$\ddot{x} = u - \gamma, \quad u = -\frac{q}{(am^*s')}, \quad \gamma = \frac{mgk}{(m^*s'^2)}$$

$$|u| \le u_0, \quad u_0 = \frac{q_0}{(am^*s')}, \quad u_0 > |\gamma| \tag{4.4}$$

The condition of complete controllability for all x, $\dot{x}$ has a clear meaning, the maximum control moment q_0 must exceed the moment of the gravitational force $mga|\sin\beta|$.

System (4.3) will be controllable for fairly small values of $|\mu'|\dot{x}^2$ for the condition $q_0 > mga|\sin\beta|$, analogous to (4.4). The function $\sin\beta(x)$, when $\lambda x_n = \pi(n+\frac{1}{2})$, reaches extremal values

$$\sin\beta_n^{\pm} = y_n'(1+y_n'^2)^{-1/2} = k(1+k^2)^{-1/2} \pm \xi(1+k^2)^{-3/2} + O(\xi^2), \quad |\xi| \ll 1 \tag{4.5}$$

The condition of controllability for arbitrary x and fairly small $|\dot{x}|$ should be verified taking into account the variability of the slope using formulae (4.5). If the variations in the slope are relatively small, controllability occurs over an asymptotically large range of variation of $\dot{x}$, $|\dot{x}| \sim |\xi|^{-1/2}$, $|\xi| \ll 1$.

Note that the strong inequality $|\xi| \ll 1$ allows of values as large as desired of the amplitude $|h|$ (for example, $|h| \sim a$ or $|h| \gg$ a) if the number $|\lambda|$ is sufficiently small ($|\lambda|a \ll 1$). Conversely, the value of $[\lambda]$ can be large ($|\lambda|a \sim 1$ or $|\lambda|a \gg 1$), if the amplitude $|h|$ is fairly small ($|h| \ll 1$). Finally the condition $|a\kappa| < 1$ must be satisfied; for this it is sufficient that $ah\lambda^2 < 1$.

Consider the case of small curvature: $|\kappa| \ll a^{-1}$ (see the corresponding relation (4.2)) and we will use the perturbation method to synthesise the optimal control for the time-optimal problem.[9–11] We will take the radius of the disc as the natural scale of length l: $l = a$, and it is convenient to relate the unit of time τ to the value of the parameter q_0 (4.3): $\tau^2 = m^*a^2(1+k^2)^{1/2}q_0^{-1}$. We will introduce the small parameter ε, which we will associate with the small quantities $\varepsilon\alpha = a\lambda$ and $\varepsilon\xi = h\lambda$. Instead of the degenerate equation (4.4) we obtain a more general perturbed equation, which enables us to take variations in the slope of the profile that are periodic in x into account. This equation can be represented in the form of the system

$$\dot{x} = \upsilon, \quad \dot{\upsilon} = u - \gamma + \varepsilon f$$
$$f(x, \upsilon, u, \varepsilon) \equiv f_g(x) + \varepsilon f_\mu(x)\upsilon^2 + f_q(x)u, \quad |u| \le 1 \tag{4.6}$$

The perturbing functions are periodic in x; they are determined by the corresponding factors: f_g is the variation of the moment of the gravitational force, f_μ is the quantity $\mu' \sim \varepsilon^2$ and f_g are the variations of the coefficients for the control q (see Eq. (4.3)).

For system (4.6) we have the two-point time-optimal problem of the type (3.9)

$$x(0) = x^0, \quad \upsilon(0) = \upsilon^0; \quad x(t_f) = x^f, \quad \upsilon(t_f) = \upsilon^f$$
$$J[u] = t_f \to \min_u, \quad |u| \le 1; \quad |\gamma| < 1, \quad |\varepsilon| \ll 1 \tag{4.7}$$

System (4.6) belongs to the class of "non-oscillating objects", for which the theory and methods for regular synthesis in a bounded region have been developed.[10] The approach to the solution is based on the construction of a switching curve; the time-optimal control is a bang-bang control and has a single switching point on this curve. An analytical method has also been developed,[11] using the perturbation method. As it applies to the problem in question, the construction of the switching curve reduces to integration of the equations for its branches, corresponding to $u = \pm 1$ and starting from the point (x^f, υ^f). The corresponding Cauchy problems have the form

$$\frac{dx}{d\upsilon} = \frac{\upsilon}{u-\gamma} - \frac{\varepsilon f \upsilon}{(u-\gamma)(u-\gamma+\varepsilon f)}; \quad x(\upsilon^f, \varepsilon) = x^f$$
$$u = +1, \quad \upsilon < \upsilon^f; \quad u = -1, \quad \upsilon > \upsilon^f; \quad |x| < \infty, \quad |\upsilon| \sim \varepsilon^{-1/2} \tag{4.8}$$

The function f is defined by (4.6). When $\varepsilon = 0$ (the unperturbed system (4.4)) the branches of the parabola $x_0^{\pm}(\upsilon)$ have a simple representation[11] (see Fig. 2, the continuous curves)

$$x_0^{\pm}(\upsilon) = x^f + \frac{\upsilon^2 - \upsilon^{f2}}{2(u-\gamma)}, \quad u = \pm 1, \quad \pm\upsilon < \upsilon^f \tag{4.9}$$

The function $x_0^{\pm}$ has a kink point (a jump of the tangent) $2\upsilon^f(1-\gamma^2)^{-1}$. The case when $\upsilon^f = 0$ is usually considered; then the branches $x^{\pm} = 0$ (4.9) are matched smoothly at the point x^f, 0 when $u = \pm 1$ (see Fig. 2, the dashed curves). The branch for $u = +1$, $\upsilon < 0$ departs downwards to the right, and upwards to the left for $u = -1$, $\upsilon > 0$ (similar to the standard classical version of the problem[9,10]). The moment of the force of gravity γ either facilitates or obstructs the action of the control action u. The minimal time is calculated in a standard way[10,11] using the known phase trajectory.

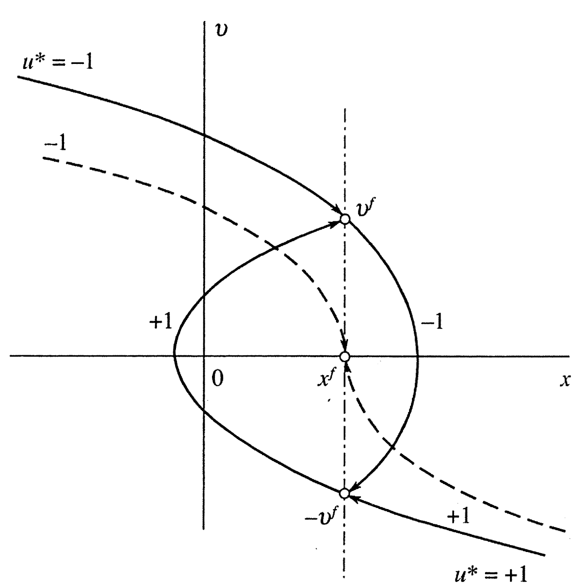

Fig. 2.

The perturbing function $f(x, v, u, \varepsilon)$ is taken into account by the perturbation method: the functions constructed at the preceding steps of the recurrence procedure[11] are substituted into it

$$x_1^{\pm}(v,\varepsilon) = x_0^{\pm}(v) - \varepsilon(u-\gamma)^{-2}\int_{v^f}^{v} zf(x_0^{\pm}(z), 0, u, 0)dz$$

$$x_{j+1}^{\pm}(v,\varepsilon) = x_0^{\pm}(v) - \varepsilon\int_{v^f}^{v} zf_*(x_j^{\pm}, z, u, \varepsilon)dz, \quad j = 1, 2, \dots$$

$$f_* \equiv f(u-\gamma)^{-1}(u-\gamma+\varepsilon f)^{-1}, \quad u = \pm 1, \quad v \gtrless v^f, \quad |v| \sim \varepsilon^{-1} \tag{4.10}$$

Here, for brevity, we have introduced the function f_*, which defines the perturbing term in Eqs (4.8). For sufficiently small ε the values of $|v|$ can be taken as large as desired up to $|v| \sim \varepsilon^{-1/2}$.

Procedure (4.10) provides a methodology but is hardly effective for practical calculations. Numerical integration of Cauchy problems (4.8) is preferable. For analytical estimates of the effect of the perturbations, i.e., curvatures of the profile, on the controlled rolling of the disc can be confined to the first and second approximations of the switching curve $x_{1,2}^{\pm}(v, \varepsilon)$ (4.10).

According to the feedback control structure[9–11] the optimal control $u = +1$ for the phase points (x, v), situated under the switching curve $x^{\pm}(v, \varepsilon)$ or on the branch $x^{+}(v, \varepsilon)$. The control $u = -1$ for points above the switching curve or on the branch $x^{-}(v, \varepsilon)$. Note that the function $x^{\pm}(v, \varepsilon)$ can be asymmetric with respect to v when $v^f \neq 0$ or $\gamma \neq 0$ (see Fig. 2). Considerable variations of the branches of the parabola of the switching curve will occur when the parameter ε increases due to the effect of the periodic components of the profile $u(x)$ (4.1).

An investigation of the feedback control structure for different values of the system parameters k, h, λ, a, m, m^*, g, q_0 requires separate detailed modelling. It includes a numerical integration of Cauchy problems (4.8) and of the equations of motion (4.6) for appropriate values of $u = \pm 1$. The optimal control implies a highly accurate determination of the switching points, i.e., intersection of the phase trajectories of the switching curve, along which the phase point moves on the final part of the trajectory. Naturally a check of the conditions for regular rolling without slipping and without loss of contract (3.8) is required.

Note that system (4.6) possesses a structure which allows of analytical integration of the equation in the phase plane (x, v). In fact, it has the form of a Bernoulli equation and reduces to a linear equation for unknown v^2. The branches of the trajectories are represented by quadratures, the calculation of some of which is difficult and requires mathematical modelling. This indicates that it is preferable from the beginning to carry out numerical integration of these Cauchy problems (4.10). In addition, if $v^f \neq 0$, there is a region of values of x for which the dependence of v on x is not single-valued (see Fig. 2). It is more convenient to construct the single-valued inverse dependence $x^{\pm}(v)$ numerically.

2°. We will consider another limiting case, when the slope of the supporting curve k is small and can be neglected $(k = 0)$. However, the periodic component is assumed to be important: the amplitude of the rise h is of the order of a and may be very large. For simplicity and physical realizability we will assume that a $\kappa < 1$, which occurs when $a\xi < 1$, according to relations (4.3). The disc is above the curve $y = h(1 - \cos\lambda x)$ and rolls under the action of the moments of gravity and control forces. In Eq. (4.3) we will assume that

$$\kappa = \left(\frac{\varepsilon\lambda}{s'^3}\right)\cos\lambda x, \quad s' = (1+\xi^2\sin^2\lambda x)^{1/2} \geq 1 \tag{4.11}$$

We will calculate the value of the path $s(x)$ travelled, taking into account the sign for a specified value of x. As was pointed out, it is determined in terms of the complete elliptic integral E and the incomplete elliptic integral F of the second kind[12] by integrating the function $s'(x)$ (4.11)

$$s(x) = \left(\frac{\xi}{\delta}\right)\left[F\left(x+\frac{\pi}{2}, \delta\right) - E(\delta)\right], \quad \delta = \xi(1+\xi^2)^{-1/2} < 1$$

$$s(x) = \begin{cases} x + \xi^2\lambda^2\dfrac{x^3}{6} + O(\lambda^4x^5), & \lambda x \ll 1 \\ x + \dfrac{x-(2\lambda)^{-1}\sin 2\lambda x}{2} + O(\xi^4 x), & \xi \ll 1 \end{cases} \tag{4.12}$$

Hence the inequality $s/x > 1$ obviously follows from (4.12) when $x \neq 0$.

We will reduce the equation of motion to dimensionless form by introducing convenient units of measurement in order to reduce the number of parameters as far as possible. Note that the initial system (4.3) contains eight parameters, defining the mass-inertial, geometrical and force characteristics. When $k = 0$, the following seven parameters remain: m, m^*, a, h, λ, g, q_0; by normalizing them the number can be reduced to three. In fact, introducing the dimensionless variable $\theta = \lambda x$, the argument $\tau = vt$ and the control u, we have a generalized pendulum type controlled system[7,11]

$$\chi\ddot{\theta} + \chi'\dot{\theta}^2 + \frac{1}{s'}\sin\theta = u, \quad |u| \le u_0$$
$$\nu^2 = \frac{m}{m^*}\frac{g}{h}\xi^2, \quad u = -\frac{q\lambda}{am^*\nu^2}, \quad s' = (1+\xi^2\sin^2\theta)^{1/2}, \quad \chi(\theta) = \left(1 - \frac{\alpha\xi\cos\theta}{s'^3}\right)s'$$
$$\xi = \lambda h, \quad \alpha = a\lambda \tag{4.13}$$

Controlled system (4.13) contains three dimensionless parameters: ξ, α, u_0; the function $\chi \ge \chi^* > 0$ by virtue of the assumption $\alpha\xi = \lambda^2 ah < 1$ (see above). When $u_0 > (1+\xi^2)^{-1/2}$ the control moment of forces exceeds the moment of the gravitational force; the system will be controllable for all x, $v = \dot{x}$. For applications the solution of time-optimal problems is of considerable interest.[9–11] The required control and trajectory can be constructed numerically or approximately analytically using the maximum principle. In particular, the switching curve and the trajectories can be obtained by constructing a regular synthesis similar to the procedure described in Section 1°.

We will consider the situation, which is of interest from theoretical and applied aspects, when the control action is small: $u_0 \ll 1$. The problem is to transfer the generalized penedulum (4.13) to the required state of motion, which is characterized by the total energy $E \sim 1$ in a minimum time τ_f.[7,11] The kinetic energy T, the potential energy V and the total energy E of the uncontrolled system ($u \equiv 0$) are found from expressions (3.2) and (3.3)

$$T = \frac{1}{2}\chi^2(\theta)\dot{\theta}^2, \quad V = 1 - \cos\theta + \frac{\alpha}{\xi}[(1+\xi^2\sin^2\theta)^{-1/2} - 1], \quad E = T + V = \text{const}$$
$$0 \le V \le V^* = 2, \quad \theta^* = (2n+1)\pi \tag{4.14}$$

If $u \neq 0$, $|u| \le u_0 \ll 1$, the oscillatory-rotatory system (4.13) is weakly controllable, and its total energy E (4.14) is a slow variable[7,11]

$$\dot{E} = \chi(\theta)\dot{\theta}u, \quad |u| \le u_0, \quad E(0) = E^0 \tag{4.15}$$

The quantity E^0 is defined in terms of θ^0, $\dot{\theta}^0$, according to expressions (4.14) for E, T and V. If during the control time $0 \le \tau \le \tau_f$ many (in practice several) oscillations or rotations of system (4.13) occur, an effective numerical-analytical procedure for constructing the approximate optimal solution[11] is applied to it. The procedure is based on the use of the averaging method for an approximate solution of the boundary-value problem of the maximum principle. According to the algorithm which has been developed, the equation of motion (4.13) is reduced to standard form in energy-phase variables.

The energy E is defined by formulae (4.14), and its evolution is defined by relations (4.15). Similar expressions for the phase ψ first require extremely lengthy calculations of the period $\Theta(E)$ in the oscillation mode Θ_v and the rotation mode Θ_r (Refs 7, 11)

$$\Theta(E) = \oint\frac{d\theta}{\dot{\theta}}, \quad \dot{\theta} = \pm\frac{\sqrt{2}(E - V(\theta))^{1/2}}{\chi(\theta)}$$
$$\Theta_\upsilon = 2\sqrt{2}\int_0^a \frac{\chi(\theta)}{(E - V(\theta))^{1/2}}d\theta$$
$$a(E) = \arg_\theta(E - V(\theta)), \quad E < 2; \quad \Theta_r = \frac{1}{\sqrt{2}}\int_0^{2\pi}\frac{\chi(\theta)}{(E - V(\theta))^{1/2}}d\theta, \quad E > 2 \tag{4.16}$$

Here a is the amplitude of the oscillations, i.e., the root of the equation $E = V(\theta)$. On approaching the separatrix ($E \to 2$) the periods of the oscillations and rotations increase without limit.

The unperturbed phase (when $u = 0$) is found in the form of a quadrature

$$\psi = \omega(E)\tau + \text{const} = \frac{\omega}{\sqrt{2}}\int\chi(\theta)(E - V(\theta))^{-1/2}d\theta, \quad \omega = \frac{2\pi}{\Theta} \tag{4.17}$$

Differentiating expression (4.17) with respect to τ, by virtue of the perturbed equation (4.13), we obtain the relation

$$\dot{\psi} = \omega(E) + \frac{1}{\sqrt{2}}\frac{\partial}{\partial E}[\omega(E)\int\chi(\theta)(E - V(\theta))^{-1/2}d\theta]\chi(\theta)\dot{\theta}(E,\theta)u \tag{4.18}$$

The right-hand sides of Eqs (4.15) for E and (4.18) for ψ are determined in terms of the initial phase variables θ, $\dot{\theta}$ using the change of variables defined by formulae (4.14) and (4.17); they also depend on E and u. In order to close the system of equations of motion in the variables E and ψ it is necessary to solve transcendental equation (4.17) for θ and substitute the function $\theta(E,\psi)$ into these right-hand sides. These equations will be 2π-periodic in ψ, regularly dependent on E and linear in u. The asymptotic method of separating the fast motions (the phase) and the slow motions (the energy and momenta), can then be applied to the corresponding boundary-value problems of the maximum principle which, in a number of cases, for example, for small (quasilinear) oscillations, considerably simplifies their solution. However, the reduction to a standard form, described above, and averaging of the equations are rather complicated, which considerably reduces the attractive properties of the general scheme of the averaging method.

For a limited formulation of the control problem, when it is required to change the value of the total energy of motion of the disc in the optimum way, the solution of the boundary-value problem in the first approximation can be constructed more simply using a modified averaging procedure. In fact, for the problem of the time-optimal required change in the energy we obtain the relations[7,11]

$$E(\tau_f) = E^f, \quad \tau_f \to \min_u, \quad |u| \le u_0$$
$$u^* = u_0 \operatorname{sign}[(E^f - E)\dot{\theta}], \quad \chi(\theta) > 0$$
$$\dot{E} = u_0 W(E) \operatorname{sign}(E^f - E), \quad E(0) = E^0 \tag{4.19}$$

The expression for E has the form (4.14). The function W is obtained by substituting the synthesis of the control u^* (4.19) into Eq. (4.15) and averaging over ψ, which is equivalent to integrating the right-hand side along the trajectory of the unperturbed system. In the oscillation and rotation modes the corresponding Θ_v and Θ_r expressions for W_v and W_r are obtained[7]

$$W(E) = \frac{1}{2\pi}\int_0^{2\pi} |\dot{\theta}_*(E,\psi)|\chi(\theta_*(E,\psi))d\psi = \frac{1}{\Theta}\oint \chi(\theta)\operatorname{sign}\dot{\theta}(E,\theta)d\theta \tag{4.20}$$

The last integral, according to system (4.13), reduces to a set of expressions for s in terms of the elliptic integrals (4.12) and an elementary quadrature. The main difficulty is calculating the values of the periods of the oscillatory and rotatory motions $\Theta = \Theta_{v,r}(E)$ (4.16). Since $\Theta_{v,r} \to \infty$ when $E \to 2$, the efficiency of the control u^* vanishes for motion along the separatrix. However, this singularity is integrable and leads to a small increase in the error of the averaging method.[7,11] The time of the motion when passing through the separatrix changes continuously (after separating the variables τ and E).

The quasi-optimal control u^* (4.19) in the disc oscillation mode (swinging up or damping) is a bang-bang control, which presents certain difficulties when modelling system (4.13) over a long time interval. Instead of a bang-bang function we propose to use a "smoothed" function, having no points of discontinuity[7]

$$u_*(\theta,\dot{\theta}) = \frac{E^f - E}{|E^f - E| + d_E}\,\frac{\dot{\theta} + c}{|\dot{\theta}| + d_\theta} \tag{4.21}$$

Here c and $d_{E,\theta}$ are the adjustment parameters, where $d_{E,\theta}$ are positive. For small $d_{E,\theta}$ and $|c|$ expression (4.21) is "close" to the function u^* (4.19) (in the integral sense) and in the limit is identical with it. It is obvious that the condition $|c| \le d_\theta$ must be satisfied, this ensures the required constraint $|u_*| \le 1$. The quantity E defines the properties of the final state θ^f, $\dot{\theta}^f$. For example, when $\theta^f = \dot{\theta}^f = 0$ (a rest point) $E^f = 0$; motion along the separatrix and, in particular, $\theta^f = \pi(2n+1)$, $\dot{\theta}^f = 0$, corresponds to the state $E^f = 2 = V^*$. The occurence of the parameter c in the function u_* simplifies the start of the motion of the wheel from the state of rest $\theta^0 = \dot{\theta}^0 = 0$. It also enables us to transfer to rotation (continuous motion) in the required direction (forward or backward).

An equation similar to (4.13) describes the controlled rolling of a non-uniform cylinder along a horizontal plane by the control motion of a flywheel. Quasi-optimal swinging and permanent rolling have been investigated in detail by numerical-analytical methods.[7]

Hence, we have constructed modes of controlling the motion of a wheel along a wavy path. To investigate the particular features of the rolling in the case of "small" (Section 2°) and "large" (Section 1°) moments of the forces, extensive mathematical modelling is required. Note that the action (forces or moments of forces) can be obtained by controlled displacement of internal masses.[5–7]

Acknowledgements

This research was financed by the Russian Foundation for Basic Research (08-01-00234) and the Programme for the Support of the Leading Scientific Schools (NSh 4315.2008.1).

References

1. Routh EJ. *Dynamics of a System of Rigid Bodies*. New York: Dover; 1905.
2. Appel P. *Traité de Mécanique Rationelle. T. 2*. Paris: Gauthier-Villars; 1953.
3. Macmillan W.D. *Dynamics of rigid bodies*. New York: McGraw-Hill, 1936.
4. Markeyev AP. *The Dynamics of a Body in Contact with a Solid Surface*. Moscow: Nauka; 1992.
5. Martynenko Yu G, Formal'skii A.M. Control of the longitudinal motion of a monocycle on an uneven surface. *Izv Ross Akad Nauk. Teoriya i Sistemy Upravleniya* 2005; 4: 165-73.
6. Martynenko YuG, Formal'skii AM. Problems of controlling unstable systems. *Uspekhi Mekhaniki* 2005;**3**(2):71–135.
7. Akulenko LD, Bolotnik NN, Kumakshev SA, Nesterov SV. Control of the motion of a non-uniform cylinder with mobile internal masses on a horizontal plane. *Prikl Mat Mekh* 2006;**70**(6):942–58.
8. Gantmakher FR. *Lectures on Analytical Mechanics*. Moscow: Nauka; 1966.
9. Pontryagin LS, Boltyansky VG, Gamkrelidze RV, Mishchenko EF. *Mathematical Theory of Optimal Processes*. New York: Wiley; 1962.
10. Boltyanskii VG. *Mathematical Methods of Optimal Control*. New York: Holt, Rinehart and Winston; 1971.
11. Akulenko LD. *Asymptotic Methods of Optimal Control*. Moscow: Nauka; 1987.
12. Gradshteyn IS, Ryzhik IM. *Tables of Integrals, Sums, Series and Products*. San Diego CA: Academic Press; 2000.

Translated by R.C.G.